Biclique-colouring powers of paths and powers of cycles*

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Abstract. Biclique-colouring is a colouring of the vertices of a graph in such a way that no maximal complete bipartite subgraph with at least one edge is monochromatic. We show that it is co$\mathcal{NP}$-complete to check if a colouring of vertices is a valid biclique-colouring, a result that justifies the search for structured classes where the biclique-colouring problem could be efficiently solved. We consider biclique-colouring restricted to powers of paths and powers of cycles. We determine the biclique-chromatic number of powers of paths and powers of cycles. The biclique-chromatic number of a power of a path P_n^k is $\max(2k+2-n, 2)$ if $n \geq k+1$ and exactly n otherwise. The biclique-chromatic number of a power of a cycle C_n^k is at most 3 if $n \geq 2k+2$ and exactly n otherwise; we additionally determine the powers of cycles that are 2-biclique-colourable. All the proofs are algorithmic and we provide polynomial-time biclique-colouring algorithms for graphs in the investigated classes.
Keywords: powers of cycles, powers of paths, hypergraphs, biclique-colouring.

1 Introduction

Let $G = (V, E)$ be a simple graph with order $n = |V|$ vertices and $m = |E|$ edges. A *clique* of G is a maximal set of vertices of size at least 2 that induce a complete subgraph of G. A *biclique* of G is a maximal set of vertices that induce a complete bipartite subgraph of G with at least one edge. A *clique-colouring* of G is a mapping that associates a colour to each vertex such that no clique is monochromatic. If the mapping uses at most k colours we say that π is a *k-clique-colouring*. A *biclique-colouring* of G is a mapping that associates a colour to each vertex such that no biclique is monochromatic. If the mapping uses at most k colours we say that π is a *k-biclique-colouring*. The *clique-chromatic number* of G, denoted by $\kappa(G)$, is the least k for which G has a k-clique-colouring. The *biclique-chromatic number* of G, denoted by $\kappa_B(G)$, is the least k for which G has a k-biclique-colouring.

Both clique-colouring and biclique-colouring have a "hypergraph colouring version". Recall that a hypergraph $\mathcal{H} = (V, \mathcal{E})$ is an ordered pair where V is a

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set of vertices and $\mathcal{E}$ is a set of hyperedges, each of which is a set of vertices. A colouring of hypergraph $\mathcal{H} = (V, \mathcal{E})$ is a mapping that associates a colour to each vertex such that no hyperedge is monochromatic. Let $G = (V, E)$ be a graph and let $\mathcal{H}_C(G) = (V, \mathcal{E}_C)$ and $\mathcal{H}_B(G) = (V, \mathcal{E}_B)$ be the hypergraphs whose hyperedges are, respectively, $\mathcal{E}_C = \{K \subseteq V \mid K \text{ is a clique of } G\}$ and $\mathcal{E}_B = \{K \subseteq V \mid K \text{ is a biclique of } G\}$ — $\mathcal{H}_C(G)$ and $\mathcal{H}_B(G)$ are called, resp., the *clique-hypergraph* and the *biclique-hypergraph* of G. A clique-colouring of G is a colouring of its clique-hypergraph $\mathcal{H}_C(G)$; a biclique-colouring of G is a colouring of its biclique-hypergraph $\mathcal{H}_B(G)$.

Clique-colouring and biclique-colouring are analogous problems in the sense that they refer to the colouring of hypergraphs arising from graphs. In particular, subsets of vertices that are maximal (in the original graph) with respect to some property — that property is "being a clique" or "being a biclique". The clique is a classical important structure in graphs, hence it is natural that the clique-colouring problem has been studied for a long time — see [1,12,21,25]. Only recently the biclique-colouring problem started to be investigated [31]. Many other problems, initially stated for cliques, have their version for bicliques [3,20], such as *Ramsey number* and *Turán's theorem*. The combinatorial game called on-line Ramsey number also has a version for bicliques. Although complexity results for complete bipartite subgraph problems are mentioned in [17] and the (maximum) biclique problem is shown to be $\mathcal{NP}$-hard in [33], only in the last decade the (maximal) bicliques were rediscovered in the context of counting problems [18,28], enumeration problems [13,27], and intersection graphs [19].

Clique-colouring and biclique-colouring have similarities with usual vertex-colouring. A proper vertex-colouring is also a clique-colouring and a biclique-colouring — in other words, both the clique-chromatic number and the biclique-chromatic number are bounded above by the vertex-chromatic number. Optimal vertex-colourings and clique-colourings coincide in the case of K_3-free graphs, while optimal vertex-colourings and biclique-colourings coincide in the (much more restricted) case of $K_{1,2}$-free graphs — notice that the triangle K_3 is the simplest complete graph larger than the graph induced by one edge (K_2), while the $K_{1,2}$ is the simplest complete bipartite graph larger than the graph induced by one edge ($K_{1,1}$). But there are also essential differences, most remarkably, it is possible that a graph has a clique-colouring (resp. biclique-colouring), which is not a clique-colouring (resp. biclique-colouring) when restricted to one of its subgraphs. Subgraphs may even have a larger clique-chromatic number (resp. biclique-chromatic number) than the original graph.

We begin this paper with a result that exhibits the difficulty of the biclique-colouring problem: even to check if a colouring of vertices is a valid biclique-colouring is a difficult task, being co$\mathcal{NP}$-complete even when the input is K_4-free. We therefore select two structured classes for which we can provide efficient biclique-colouring algorithms: powers of paths and powers of cycles. The choice of those classes has strong motivation since they have been recently investigated in the context of well studied variations of colouring problems. For instance, Effantin and Kheddouci [14] proved, for a power of a path P_n^k, that its b-chromatic

number is n, if $n \leq k+1$; $k+1+\lfloor\frac{n-k-1}{3}\rfloor$, if $k+2 \leq n \leq 4k+1$; or $2k+1$, if $n \geq 4k+2$. They also proved, for a power of a cycle C_n^k, that its b-chromatic number is n, if $n \leq 2k+1$; $k+1$, if $n = 2k+2$; at least $\min(n-k-1, k+1+\lfloor\frac{n-k-1}{3}\rfloor)$, if $2k+3 \leq n \leq 3k$; $k+1+\lfloor\frac{n-k-1}{3}\rfloor$, if $3k+1 \leq n \leq 4k$; or $2k+1$, if $n \geq 4k+2$. Moreover, other well studied variations of colouring problems when restricted to powers of cycles have been investigated: chromatic number [29], chromatic index [26], total chromatic number [9], choice number [29], and clique-chromatic number [8]. It is known, for a power of a cycle C_n^k, that the chromatic number and the choice number are both $k+1+\lceil r/q \rceil$, where $n = q(k+1)+r$ with $q \geq 1$, $0 \leq r \leq k$ and $n \geq 2k+1$, the chromatic index is the maximum degree of C_n^k if and only if n is even, the total chromatic number is at most the maximum degree of C_n^k plus 2, when n is even and $n \geq 2k+1$, and the clique-chromatic number is 2, when $n \leq 2k+1$, and is at most 3, when $n \geq 2k+2$. Particularly, in the latter case, the clique-chromatic number is 3, when n is odd and $n \geq 5$; otherwise, it is 2. Note that total colouring is an open and difficult problem and remains unsolved for powers of cycles [9]. Other significant works have been done in powers of certain classes of graphs [7,10] and, in particular, in powers of cycles [5,6,22,23,24,32].

2 Complexity of biclique-colouring

The biclique-colouring problem is a variation of the clique-colouring problem. Hence, it is natural to investigate the complexity of biclique-colouring based on the tools that were developed to determine the complexity of clique-colouring. In the present work we show that, similarly to the case of clique-colouring, it is co$\mathcal{NP}$-complete to check if a vertex colouring is a valid biclique-colouring. The co$\mathcal{NP}$-completeness holds even when the input is K_4-free.

Theorem 1. *Given a K_4-free graph G and a function $\pi : V(G) \to \{1, ..., k\}$, it is co$\mathcal{NP}$-Complete to check if π is a k-*BICLIQUE-COLOURING.

The proof is a reduction of the 3-SAT problem to the BICLIQUE CONTAINMENT problem (which is checking if there exists a biclique of a graph G contained in a given subset of the vertices of G), the complement of the problem of checking if a vertex colouring is a valid biclique-colouring. The details can be found in the appendix.

3 Powers of paths, powers of cycles, and their bicliques

A *power of a path* P_n^k is a simple graph with $V(G) = \{v_0, \ldots, v_{n-1}\}$ and $\{v_i, v_j\} \in E(G)$ if, and only if, $|i-j| \leq k$. Note that P_n^1 is the induced path on n vertices and P_n^k, $n \leq k+1$, is the complete graph K_n on n vertices. In a power of a path, if $e = \{v_i, v_j\} \in E(G)$ and $|i-j| = \ell$, for some $1 \leq \ell \leq k$, then edge e has *reach* ℓ and we denote $d(i,j) = \ell$. A *power of a cycle* C_n^k is a simple graph with $V(G) = \{v_0, \ldots, v_{n-1}\}$ and $\{v_i, v_j\} \in E(G)$ if, and only if,

$\min\{(j-i) \bmod n, (i-j) \bmod n\} \leq k$. Note that C_n^1 is the induced cycle on n vertices and C_n^k, $n \leq 2k+1$, is the complete graph K_n on n vertices. In a power of a cycle, we take $(v_0, \ldots, v_{n-1})$ to be a *cyclic order* on the vertex set of G and always perform arithmetic modulo n on vertex indexes. We say $\{v_i, v_j\}$ has *left reach* (resp. *right reach*) ℓ, for some $1 \leq \ell \leq k$, if $(i-j) \bmod n = \ell$ (resp. $(j-i) \bmod n = \ell$) and we denote by $\overleftarrow{d(i,j)}$ (resp $\overrightarrow{d(i,j)}$). If $e = \{v_i, v_j\} \in E(G)$ and $\min\{\overleftarrow{d(i,j)}, \overrightarrow{d(i,j)}\} = \ell$, for some $1 \leq \ell \leq k$, then edge e has *reach* ℓ and we denote $d(i,j) = \ell$.

Throughout this work, when it is not clear by the context what reach we are dealing with, we specify if it is in the context of either power of a path or power of a cycle. The definition of reach is extended to an induced path to be the sum of the reach of its edges. A P-*block* is a maximal set of consecutive vertices satisfying a property P. The *size* of a P-block is the number of vertices in the P-block. In what follows, we explicitly identify the bicliques of a power of a path and the bicliques of a power of a cycle. The extreme values are well known: $\kappa_B(K_n) = n$, $\kappa_B(P_n) = \kappa_B(C_n) = 2$. Notice that, for each range of n, every biclique in Lemmas 1 and 2 always exists. Henceforth, refer to the appendix for the omitted proofs.

Lemma 1. *The bicliques of a power of a path P_n^k are precisely: P_2 bicliques, if $n \leq k+1$; P_2 bicliques and P_3 bicliques, if $k+2 \leq n \leq 2k$; and P_3 bicliques if $n \geq 2k+1$.*

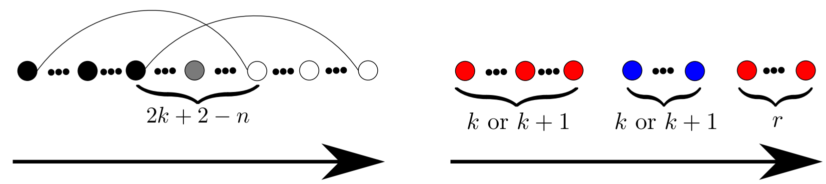
Lemma 2. *The bicliques of a power of a cycle C_n^k are precisely: P_2 bicliques, if $n \leq 2k+1$; $K_{2,2}$ bicliques, if $2k+2 \leq n \leq 3k+1$; P_3 bicliques and $K_{2,2}$ bicliques, if $3k+2 \leq n \leq 4k$; and P_3 bicliques, if $n \geq 4k+1$.*

4 Determining the biclique-chromatic number of P_n^k

In the present section, we determine the biclique-chromatic number of powers of paths. Recall that a power of a path P_n^k with $n \leq k+1$ is a complete graph whose biclique-chromatic number is its order n. We consider other two cases: $n \in [k+2, 2k]$ and $n \in [2k+1, \infty)$.

Theorem 2. *A power of a path P_n^k, when $k+1 \leq n \leq 2k$, has biclique-chromatic number $2k+2-n$.*

Sketch. In this case every pair of vertices in the sequence $(v_{n-k-1}, \ldots, v_k)$ induces a P_2 biclique in the graph. Moreover, this sequence has all P_2 bicliques of the graph, i.e. the remaining bicliques are induced P_3 bicliques left in the set of all bicliques in the graph. Hence we are forced to give distinct colours to every vertex in the sequence $(v_{n-k-1}, \ldots, v_k)$, but we can repeat an used colour in the uncoloured vertices before v_{n-k-1} and another used colour in the uncoloured vertices after v_k. We refer to Fig. 1a to illustrate the given $(2k+2-n)$-biclique-colouring. $\square$



(a) $(2k+2-n)$-biclique-colouring, when $n \leq 2k+1$.

(b) 2-biclique-colouring, when $n \geq 2k+2$ and $0 \leq r \leq k$.

Fig. 1. Biclique-colouring of powers of paths

Theorem 3. *A power of a path P_n^k, when $n \geq 2k+1$, has biclique-chromatic number 2.*

Sketch. In this case the following colouring is a valid 2-biclique-colouring: monochromatic-blocks of size calculated by the minimum between k and the number of uncoloured vertices, switching colours *red* and *blue* alternately. We refer to Fig. 1b to illustrate the given 2-biclique-colouring. $\square$

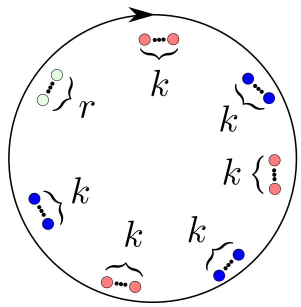
In what follows, we switch our aim from powers of paths to powers of cycles. The reader should notice the structure differences between the two classes of graphs and observe the similarities on giving lower and upper bounds on the biclique-chromatic number. For instance, the lower bound on the biclique-chromatic number in both cases when $n \leq 2k$ is a consequence of the existence of a set of K_2 bicliques whose union induces a complete graph — in the case of powers of a cycle, such union is the whose vertex set, but in the case of power of a path this is not necessarily true. When $n \geq 2k+1$ the key step to construct optimal colourings was the definition of monochromatic-blocks of size k or $k+1$. Nevertheless, in the given colourings, for a power of a path, the last vertices and the first vertices according to the cyclic order may have the same colour, which is not the case for a power of a cycle.

5 Determining the biclique-chromatic number of C_n^k

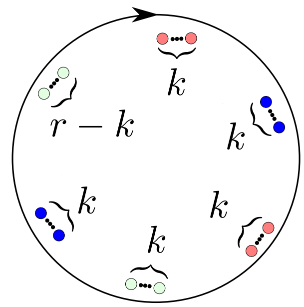
In the present section, we determine the biclique-chromatic number of powers of paths. Recall that a power of a cycle C_n^k with $n \leq 2k+1$ is a complete graph and its biclique-chromatic number is equal to its order n. Hence we need to consider only the case $n \geq 2k+2$. First we show that all such graphs are 3-biclique-colourable — the proof of Theorem 4 additionally yields an efficient 3-biclique-colouring algorithm.

Theorem 4. *A power of a cycle C_n^k, when $n \geq 2k+2$, has biclique-chromatic number at most 3.*

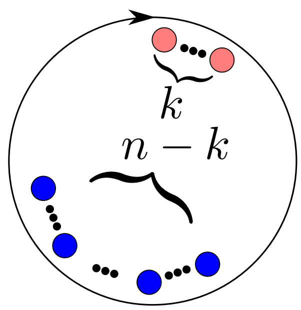
Sketch. We consider two cases depending on the remainder r of the integer division n/k: $0 < r \leq k$ and $k < r < 2k$. The corresponding colourings are shown in Figures 2a and 2b. $\square$



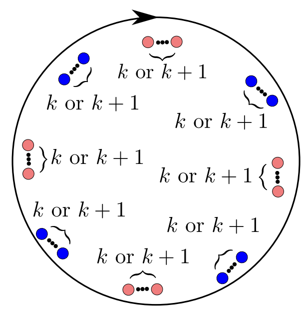
(a) 3-biclique-colouring, when $n \geq 2k+2$ and $0 \leq r \leq k$.



(b) 3-biclique-colouring, when $n \geq 2k+2$ and $k < r < 2k$.



(c) 2-biclique-colouring when $2k+2 \leq n \leq 3k+1$



(d) 2-biclique-colouring, when $n \geq 3k+2$

Fig. 2. Biclique-colouring of powers of cycles

By Theorem 4 we know that every power of a cycle, with $n \geq 2k+2$, has biclique-chromatic number 2 or 3. A natural question is to determine when $\kappa_B = 2$ and when $\kappa_B = 3$. We settle this question next and give efficient algorithms that return both the biclique-chromatic number and the optimal biclique-colouring of a power of a cycle. We consider two cases: $n \in [2k+2, 3k+1]$ and $n \in [3k+2, \infty)$. In the case $n \in [2k+2, 3k+1]$ it is always possible to obtain a 2-biclique-colouring, as stated in Theorem 5.

Theorem 5. *A power of a cycle C_n^k, when $2k+2 \leq n \leq 3k+1$, has biclique-chromatic number 2.*

Sketch. A biclique-colouring $\pi : V(G) \to \{blue, red\}$ of $G = C_n^k$ is given as follows: a monochromatic-block of size k with colour *red* followed by a monochromatic-block of size $n-k$ with colour *blue*. We refer to Fig. 6 to illustrate the given 2-biclique-colouring. $\square$

The case $n \geq 3k+2$ is more tricky (in addition to the fact that it is 3-biclique-colourable): as we show in Theorem 6, a 2-biclique-colouring will exist

if and only if every monochromatic-block has size k or $k+1$. The key step for this result is Lemma 3.

Lemma 3. *Let $G = C_n^k$, $n \geq 3k+2$, be a power of a cycle and let $\pi : V(G) \to \{blue, red\}$ be a 2-colouring of the vertices of G. Graph G has* ***no*** *monochromatic induced P_3 with reach at most $k+2$ if and only if every monochromatic-block has size k or $k+1$.*

Proof. First, suppose that all the monochromatic-blocks have size k or $k+1$. Each of the edges of G is of one of the two following types:

- an edge between two vertices belonging to the same monochromatic-block (and having the same colour); or
- an edge between two vertices belonging to consecutive monochromatic-blocks (and having distinct colours).

Hence, if we consider any three vertices v_i, v_j and v_ℓ having the same colour, then either they are in the same monochromatic-block — and induce a triangle — or they belong to at least two non-consecutive monochromatic-blocks – and induce a disconnected graph. In neither case, they can induce a P_3 and, in particular, a P_3 of reach at most $k+2$.

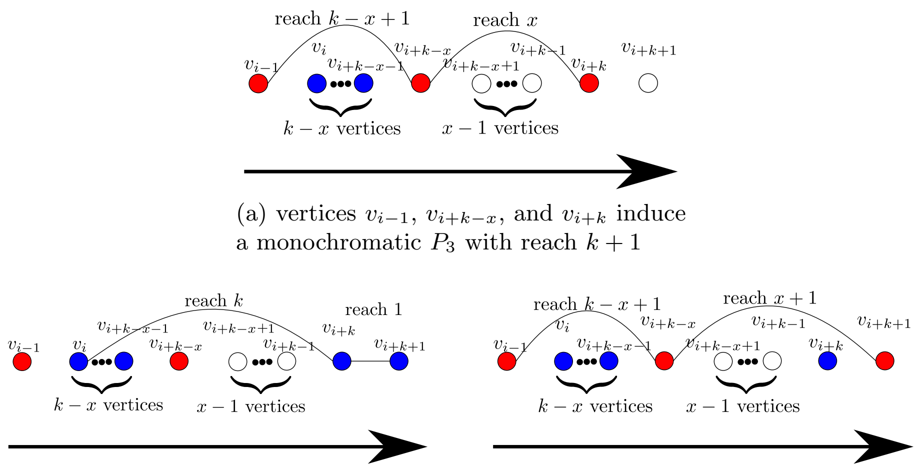
For the converse, suppose that there exists a monochromatic-block whose size is neither k nor $k+1$. If a monochromatic-block has size $p \geq k+2$, say composed of vertices $\{v_i, v_{i+1}, v_{i+2}, \ldots, v_{i+k+1}, \ldots, v_{i+p-1}\}$, then vertices $\{v_i, v_{i+1}, v_{i+k+1}\}$ induce a P_3. So, we may assume that there exists a blue-block with $k-x$ vertices, $x > 0$. We denote by v_i the leftmost vertex of the monochromatic-block — hence the rightmost vertex is $v_{i+k-x-1}$. Note that vertices v_{i-1} and v_{i+k-x} are adjacent and coloured red. Please refer to Figure 3. We consider the following cases.

- vertex v_{i+k} is coloured red. In this case, vertices v_{i-1}, v_{i+k-x} and v_{i+k} induce a monochromatic P_3 (note that vertex v_{i+k} is not adjacent to vertex v_{i-1}) with reach $k+1$. This case is depicted in Fig. 3a.
- vertex v_{i+k} is coloured blue. We consider vertex v_{i+k+1} and two subcases.
  - vertex v_{i+k+1} is coloured blue. In this case, vertices v_{i+k}, v_{i+k+1} and v_i induce a monochromatic P_3 (note that vertex v_{i+k+1} is not adjacent to vertex v_i) with reach $k+1$. This case is depicted in Fig. 3b.
  - vertex v_{i+k+1} is coloured red. In this case, vertices v_{i-1}, v_{i+k-x} and v_{i+k+1} induce a monochromatic P_3 (note that vertex v_{i+k+1} is not adjacent to vertex v_{i-1}, but is adjacent to vertex v_{i+k-x} because $x < k$) with reach $k+2$. This case is depicted in Fig. 3c. □

Theorem 6. *A power of a cycle C_n^k, when $n \geq 3k+2$, has biclique-chromatic number 2 if and only if there exist integers a and b, such that $n = ak + b(k+1)$ and $a+b \geq 2$ is even.*

Sketch. By Lemma 3 any biclique-colouring has monochromatic-blocks of sizes k of $k+1$. Numbers a and b, if they exist, give the number of monochromatic-blocks of size k and $k+1$, respectively, in a 2-biclique-colouring. □

There exists an efficient algorithm that verifies if the system of equations has a solution and, if so, computes values of a and b (see the appendix).



(a) vertices v_{i-1}, v_{i+k-x}, and v_{i+k} induce a monochromatic P_3 with reach $k+1$

(b) vertices v_i, v_{i+k}, and v_{i+k+1} induce a monochromatic P_3 with reach $k+1$

(c) vertices v_{i-1}, v_{i+k-x}, and v_{i+k+1} induce a monochromatic P_3 with reach $k+2$

Fig. 3. A monochromatic-block whose size is neither k nor $k+1$ in a power of a cycle C_n^k, with $n \geq 3k+2$, implies a monochromatic P_3 with reach at most $k+2$.

6 Final considerations

We considered the biclique-colouring problem restricted to powers of paths and powers of cycles. The classes hold natural similarities due to their construction based on vertex-orderings. However, the fact that powers of cycles reflects a cyclic (opposed to a linear) vertex-ordering leads to a higher difficulty regarding the "case analysis" — in particular, the cyclic vertex-ordering allows the appearance of $K_{2,2}$ bicliques. Table 1 highlights the exact values for the biclique-chromatic number of power graphs settled in this work. One can check, as a corollary of Theorem 6, that every power of cycle C_n^k, with $n \geq k(k+1)$, has biclique-chromatic number 2. Thus, the biclique-chromatic number of a power of a cycle C_n^k, with $n \geq 3k+2$, does not always oscillate for fixed value of k and increasing n.

Table 1. Biclique-chromatic number of some power graphs

Power graph	Range of n	Biclique-chromatic number
P_n^k	$[1, k+1]$	n
	$[k+2, 2k]$	$2k+2-n$
	$[2k+1, \infty)$	2
C_n^k	$[1, 2k+1]$	n
	$[2k+2, 3k+1]$	2
	$[3k+2, \infty)$	2, if there exist integers a and b, such that $n = ak + b(k+1)$ and $a+b \geq 2$ is even; 3, otherwise.

A *circulant graph* $C_n(d_1, \ldots, d_k)$ is a simple graph with $V(G) = \{v_0, \ldots, v_{n-1}\}$ and $E(G) = E^{d_1} \cup \cdots \cup E^{d_k}$, with $\{v_i, v_j\} \in E^{d_l}$ if, and only if, it has reach – in the context of a power of a cycle – d_l. Notice that a circulant graph $C_n(d_1, \ldots, d_k)$ is a power of a cycle if $d_1 = 1$, $d_i = d_{i-1} + 1$, $d_k < \lfloor \frac{n}{2} \rfloor$. A *distance graph* $P_n(d_1, \ldots, d_k)$ has the same definition as the circulant graph, except by the reach, which in turn is in the context of a power of a path. Notice that a distance graph $P_n(d_1, \ldots, d_k)$ is a power of a path if $d_1 = 1$, $d_i = d_{i-1} + 1$, $d_k < n - 1$. Circulant graphs have been proposed for various practical applications [4]. We consider, as a future work, to biclique colour the classes of circulant graphs and distance graphs, since colouring problems for circulant graphs and for distance graphs have been extensively investigated [2,30,34]. Moreover, some results of intractability were obtained, e.g. determining the chromatic number of circulant graphs in general is an $\mathcal{NP}$-hard problem [11].

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A Proof of Theorem 1

To achieve a result in this direction, we will take a look at the complement of the problem we are dealing with: to show a biclique in a subset of vertices of a graph G that it is also a biclique in G. We call this problem Biclique Containment problem and we introduce it formally.

Problem A1 Biclique Containment
Input: *Graph $G = (V, E)$ and $T_B \subseteq V$*
Output: *Does there exist a biclique K_B of G such that $K_B \subseteq T_B$?*

In order to show that Biclique Containment is $\mathcal{NP}$-Complete, we will use a reduction from 3-SAT problem, as follows.

Theorem 7. *The* Biclique Containment *problem is $\mathcal{NP}$-Complete, even if the input graph is K_4-free.*

Proof. Deciding whether a graph has a biclique in a given subset of vertices is in $\mathcal{NP}$: a biclique is a certificate and veryifing this certificate is trivially polynomial.

We prove that Biclique Containment problem is $\mathcal{NP}$-hard by reducing 3SAT to it. The outline of the proof follows: for every formula ϕ, a graph G is constructed with a subset of vertices denoted by T_B, such that ϕ is satisfiable if, and only if, there exists a biclique in T_B that it is also a biclique in G.

We define the graph G as follows.

- For each variable $x_i, 1 \leq i \leq n$, there exist two adjacent vertices x_i and $\overline{x_i}$.
- There exists a vertex v adjacent to x_i and $\overline{x_i}$, for every $1 \leq i \leq n$.
- For each clause $c_j, 1 \leq j \leq m$, there exists a vertex c_j. Moreover, each c_j is adjacent to a vertex $l \in \{x_1, \ldots, x_n, \overline{x_1}, \ldots, \overline{x_n}\}$ if, and only if, the literal correspondent to l is not in the clause correspondent to vertex c_j.

We define the subset of vertices T_B as $\{v, x_1, \ldots, x_n, \overline{x_1}, \ldots, \overline{x_n}\}$. Refer to Figure 4 for an example of such construction, given a formula $\phi = (x_1 \vee \overline{x_2} \vee x_4) \wedge (x_2 \vee \overline{x_3} \vee \overline{x_5}) \wedge (x_1 \vee x_4 \vee x_5)$.

We claim that formula ϕ is satisfiable if, and only if, there exists a biclique in T_B that is also a biclique in G.

For each biclique K_B of $G[T_B]$, assign a valuation v_{K_B} to formula ϕ, where variable x_i receives the true value if, and only if, the correspondent vertex is in K.

Notice that we can have two assumptions.

- A variable and its negation does not appear in the same clause. On the contrary, any assignment of values (true or false) to such a variable satisfies the clause.
- A variable appears in at least one clause. On the contrary, any assignment of values (true or false) to such a variable is indifferent to formula ϕ.

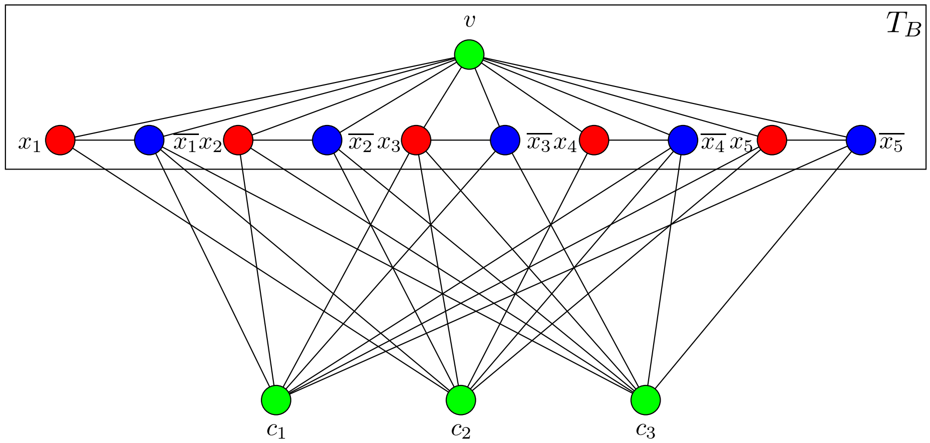

Fig. 4. Example for $\phi = (x_1 \vee \overline{x_2} \vee x_4) \wedge (x_2 \vee \overline{x_3} \vee \overline{x_5}) \wedge (x_1 \vee x_4 \vee x_5)$

We have two cases of bicliques in $G[T_B]$.

1. It does not contain vertex v. Then, the biclique is precisely formed by a pair of vertices x_i and $\overline{x_i}$, $1 \leq i \leq n$. We claim that this biclique is not a biclique in G. Suppose that it is a biclique in G. Then, for each clause c_j, $\{c_j, x_i\}$ and $\{c_j, \overline{x_i}\}$ are either both edges in G or both non-edges in G. If both are non-edges, it means, by the construction of G, that both x_i and $\overline{x_i}$ are in the same clause c_j, a contradiction with our assumptions. Then, x_i and $\overline{x_i}$ are adjacent to every $c_j, 1 \leq i \leq n$, and it means, by construction of G, that both x_i and $\overline{x_i}$ are not in any clause c_j, for every j, again a contradiction with our assumptions. In both cases, we arrived in cases that do not exist.
2. It contains vertex v. Then, the biclique is precisely formed by vertex v and exactly one vertex of each pair x_i and $\overline{x_i}$, for every $1 \leq i \leq n$. We claim that $K_B \subseteq G[T_B]$ and it is also a biclique in G if, and only if, v_{K_B} satisfies ϕ. In fact, K_B is a biclique in G if, and only if, we can not include any $c_j, 1 \leq j \leq m$, and $K_B \cup \{c_j\}$ still a biclique. Therefore, for every c_j, there not exists an edge between c_j and a vertex in K correspondent to a literal with true value in ϕ. By the definition of graph G, it is equivalent to state that every clause has at least one true literal, i.e. v_{K_B} satisfies ϕ.

$\square$

Thus, given a graph G, it is co$\mathcal{NP}$-Complete to check if a given function $\pi : V(G) \rightarrow \{1, ..., k\}$ is a k-BICLIQUE-COLOURING and it concludes the proof of Theorem 1.

B Proof of Lemma 1

Proof of Lemma 1. One can easily check that a power of a path is $K_{1,3}$-free and $K_{2,2}$-free. Thus, the bicliques of a power of a path are either P_2 or P_3.

Now, let P_n^k be a power of a path with $n \leq k+1$. We claim that always exists only P_2 biclique. In fact, since every three distinct vertices induce a K_3, the bicliques are P_2.

Now, let P_n^k be a power of a path with $k+2 \leq n \leq 2k$. We claim that always exists a P_2 biclique. Every pair of vertices in the sequence $(v_{n-k-1}, \ldots, v_k)$ are adjacent and also induces a P_2 biclique in the graph, since every vertex in that sequence is adjacent to any other vertex of P_n^k, i.e. a pair of vertices in that sequence and any other vertex of P_n^k induce a K_3. Now, we claim that always exists a P_3 biclique. A vertex in the sequence $(v_{n-k-1}, \ldots, v_k)$ is adjacent to v_1 and v_n, but v_1 is not adjacent to v_n, i.e. v_1, v_n, and a vertex in that sequence induce a P_3.

Now, let P_n^k be a power of a path with $n \geq 2k+1$. We claim that always exists only P_3 biclique. Let v_i and v_j be two adjacent vertices in P_n^k, such that $i < j$. If $j \leq k$, v_i, v_j, v_{j+k} induce a P_3, since v_i is not adjacent to v_{j+k}. Otherwise, i.e. $j \geq k+1$, $v_{j-(k+1)}, v_i, v_j$ induce a P_3, since $v_{j-(k+1)}$ is not adjacent to v_j. We conclude that every P_2 is contained in a P_3, i.e. every biclique in P_n^k is a P_3. □

C Proof of Lemma 2

Notice that we interchangeably denote by $K_{2,2}$ or C_4 a biclique whose bipartition have both size two, since $K_{2,2}$ and C_4 are isomorphic. One can easily check that a power of a cycle is $K_{1,3}$-free. Thus, the bicliques of a power of a cycle are either P_2, P_3 or $K_{2,2}$. Now, let C_n^k be a power of a cycle with $n \leq 2k+1$. Since every three distinct vertices induce a K_3, the bicliques are P_2. Otherwise, i.e. $n \geq 2k+2$, one can easily check that every P_2 is properly contained in an induced P_3. Thus, in the henceforth proofs, each biclique is either P_3 or $K_{2,2}$. Lemmas 4, 5, and 6 conclude the proof.

□

Lemma 4. *There always exists a $K_{2,2}$ biclique in a power of a cycle C_n^k with $2k+2 \leq n \leq 4k$.*

Proof. First, one can check that $k+1 \leq \lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil \leq 2k$ and if two distinct vertices have reaches either $\lfloor \frac{n}{2} \rfloor$ or $\lceil \frac{n}{2} \rceil$, then both vertices induce a P_3 with some other vertex.

Let G be a power of a cycle C_n^k with $2k+2 \leq n \leq 4k$. Let $P = \{v_h, v_s, v_r\}$ be an induced P_3 in G, such that the non-adjacent vertices have reach $\lfloor \frac{n}{2} \rfloor$. W.l.o.g, suppose P is in cyclic order, v_h and v_r are not adjacent, v_h and v_s are adjacent, and v_s and v_r are adjacent. We claim that P is properly contained in an induced C_4 and we suppose the contrary, by contradiction. Thus, there is no vertex v_c in G, such that $G[V(P) \cup \{v_c\}]$, say G', is an induced C_4. We suitably choose v_c to arrive in a contradiction: v_c is an "antipodal" vertex of v_s in the cyclic order of G, i.e. the right reach $\overrightarrow{d(c,s)}$ differs from the left reach $\overleftarrow{d(c,s)}$ by at most 1, then the reach $d(c,s)$ is $\lfloor \frac{n}{2} \rfloor$. We have two cases.

– Subgraph G' has a triangle as an induced subgraph. In what follows, we prove that we have the right reach $\overrightarrow{d(c,s)}$ and the left reach $\overleftarrow{d(c,s)}$ at most either k or $k+1$, but both cannot be $k+1$. Let $\{v_h, v_s, v_r, v_c\}$ be the cyclic order of G'. If $\{v_s, v_r, v_c\}$ induces a triangle, then the right reach $\overrightarrow{d(s,r)}$ plus the right reach $\overrightarrow{d(r,c)}$ is at most k. Otherwise, i.e. $\{v_s, v_r, v_c\}$ does not induces a triangle and the right reach $\overrightarrow{d(s,r)}$ plus the right reach of $\overrightarrow{d(r,c)}$ is at most $k+1$. The case whether $\{v_c, v_h, v_s\}$ induces (or not) a triangle is analogous. In any possible case, since G' has at least one triangle as an induced subgraph, both left reach $\overleftarrow{d(c,s)}$ and right reach $\overrightarrow{d(c,s)}$ cannot be at least $k+1$. Since n is equal to the sum of the right reaches $\overrightarrow{d(h,s)}$, $\overrightarrow{d(s,r)}$, $\overrightarrow{d(r,c)}$, and $\overrightarrow{d(c,h)}$, we conclude that the graph has at most $2k+1$ vertices, a contradiction.
– Otherwise, i.e. the right reach $\overrightarrow{d(r,c)}$ is at least $k+1$ or the right reach of $\overrightarrow{d(c,h)}$ is at least $k+1$. Suppose that the right reach $\overrightarrow{d(r,c)}$ is at least $k+1$ and make a stronger choose of v_c: the right reach $\overrightarrow{d(c,s)}$ is greater than $\overleftarrow{d(c,s)}$, i.e. right reach $\overrightarrow{d(c,s)}$ is $\lceil \frac{n}{2} \rceil$. Then, order $n = \overrightarrow{d(h,s)} + \overrightarrow{d(s,r)} + \overrightarrow{d(r,c)} + \overrightarrow{d(c,h)} = \overrightarrow{d(c,s)} - \overrightarrow{d(h,s)} + \overrightarrow{d(h,r)} + \overrightarrow{d(r,c)} \geq \lceil \frac{n}{2} \rceil - k + \lfloor \frac{n}{2} \rfloor + k + 1 = n+1$, a contradiction. The case where the left reach $\overleftarrow{d(r,c)}$ is at least $k+1$ is analogous.

□

Lemma 5. *There does not exist a P_3 biclique in a power of a cycle C_n^k with $2k+2 \leq n \leq 3k+1$.*

Proof. Let G be a power of a cycle C_n^k with $2k+2 \leq n \leq 3k+1$. Let $P = \{v_h, v_s, v_r\}$ be an induced P_3 in G. W.l.o.g, suppose P is in cyclic order, v_h and v_r are not adjacent, v_h and v_s are adjacent, and v_s and v_r are adjacent. We claim that P is properly contained in an induced C_4 and we suppose the contrary, by contradiction. Thus, there not exists a vertex v_c in G, such that $G[V(P) \cup \{v_c\}]$, say G', is an induced C_4. We suitably choose v_c to arrive in a contradiction: v_c is an "antipodal" vertex of v_s in the cyclic order of G, i.e. the right reach $\overrightarrow{d(c,s)}$ differs from the left reach $\overleftarrow{d(c,s)}$ by at most 1, then the reach $d(c,s)$ is $\lfloor \frac{n}{2} \rfloor$. Vertices v_c and v_s are not adjacents, since their reach is $\lfloor \frac{n}{2} \rfloor$ and $\lfloor \frac{n}{2} \rfloor \geq k+1$. Moreover, order $n = \overrightarrow{d(h,r)} + \overrightarrow{d(r,h)} \geq k+1+\overrightarrow{d(r,h)} \leftrightarrow \overrightarrow{d(r,h)} \leq n-k-1 \leq 2k$. Then, there exists an edge between v_r and v_c and an edge between v_c and v_s. We arrived in a contradiction, since we proved that G' induces a C_4.

□

Lemma 6. *There always exists a P_3 biclique in a power of a cycle C_n^k with $n \geq 3k+2$.*

Proof. Let $n \geq 2k+2$, one can easily check that every P_2 is properly contained in an induced P_3. Thus, each biclique is either P_3 or $K_{2,2}$.

Now, let G be a power of a cycle C_n^k with $n \geq 3k+2$. Let $P = \{v_h, v_s, v_r\}$ be an induced P_3 in G with reach $k+1$. W.l.o.g, suppose P is in cyclic order, v_h and v_r are not adjacent, v_h and v_s are adjacent, and v_s and v_r are adjacent. We claim that P is a biclique, i.e. there is no vertex v_c in G, such that $G[V(P) \cup \{v_c\}]$ is not an induced C_4 and we suppose the contrary, by contradiction. Then, order $n = \overrightarrow{d(h,s)} + \overrightarrow{d(s,r)} + \overrightarrow{d(r,c)} + \overrightarrow{d(c,h)} \leq k+1 + \overrightarrow{d(r,c)} + \overrightarrow{d(c,h)} \leq 3k+1$, a contradiction.

□

D Proof of Theorem 2

Proof of Theorem 2. Let $G = P_n^k$, $n \leq 2k$, be a power of a path. Every pair of vertices in the sequence $(v_{n-k-1}, \ldots, v_k)$ induces a P_2 biclique in the graph. Hence, we are forced to give distinct colours to every vertex in the sequence $(v_{n-k-1}, \ldots, v_k)$ and $\kappa_B(G) \geq 2k+2-n$.

We define $\pi : V(G) \to \{1, \ldots, 2k+2-n\}$ as follows: give (arbitrarly) distinct colours $1, \ldots, 2k+2-n$ to vertices $v_{n-k-1}, \ldots, v_k$. Now, repeat an used colour, say 1, in the uncoloured vertices before v_{n-k-1} and another used colour, say 2, in the uncoloured vertices after v_k. The remaining bicliques, if some, are induced P_3 bicliques with a vertex before v_{n-k-1} and a vertex after v_k. By the given colouring, all P_3 bicliques are polychromatic, then $\kappa_B(G) \leq 2k+2-n$.

We refer to Fig. 1a to illustrate the given $(2k+2-n)$-biclique-colouring.

□

E Proof of Theorem 3

Proof of Theorem 3. Let $G = P_n^k$, $n \geq 2k+1$, be a power of a path. We define $\pi : V(G) \to \{blue, red\}$ as follows: a number n/k of monochromatic-blocks of size k switching colours *red* and *blue* alternately, followed by a monochromatic-block of size $n \bmod k$ with colour *red*, if n/k is even or colour *blue* if n/k is odd. We refer to Fig. 1b to illustrate the given 2-biclique-colouring.

By Lemma 1, every biclique is an induced P_3. Thus, every biclique is polychromatic, since they contain vertices from two distinct but consecutive monochromatic-blocks (with distinct colours by the given colouring).

□

F Further comments on Section 5

A very basic number theory technique is occasionally used to give a biclique-colouring. The division algorithm, outlined below, is used to show that any natural number a can be expressed using the equality $a = bq+r$, with a requirement that $0 \leq r < q$, as follows.

Theorem 8 (Division algorithm). *Given two natural numbers a and b, with $b \neq 0$, there exist unique natural numbers q and r such that $a = bq + r$ and $0 \leq r < b$.*

We need a minor modification where b is even and r is strictly less than $2k$.

Corollary 1. *Given two natural numbers n and k, with $n \geq 2k+2$. There exist natural numbers a and r such that $n = ak + r$, $a \geq 2$ is even, and $0 \leq r < 2k$.*

Proof of Theorem 3. Let $a = 2x$ with $x \in \mathbb{N}_+$, since a is even. Then, $n = 2xk + r = x(2k) + r$ and by Theorem 8, we are done. $\square$

G Proof of Theorem 4

Proof of Theorem 4. Let $G = C_n^k$, $n \geq 2k+2$, be a power of a cycle. Let $V(G) = \{v_0, \ldots, v_{n-1}\}$. By Corollary 1, $n = ak + r$ for integers a and r, even $a \geq 2$, and $0 \leq r < 2k$. If $0 \leq r \leq k$, we define $\pi : V(G) \to \{blue, red, green\}$ as follows: an even number a of monochromatic-blocks of size k switching colours *red* and *blue* alternately, followed by a monochromatic-block of size r with colour *green*. Otherwise, i.e. $k < r < 2k$, we define $\pi : V(G) \to \{blue, red, green\}$ as follows: an odd number $a - 1$ of monochromatic-blocks of size k switching colours *red* and *blue* alternately, followed by a monochromatic-block of size k with colour *green*, a monochromatic-block of size k with colour *blue*, and a monochromatic-block of size $r - k$ with colour *green*. We refer to Fig. 2a to illustrate the former 3-biclique-colouring and to Fig. 2b to illustrate the latter 3-biclique-colouring.

Suppose that there exists a monochromatic P_3. Let B and B' be distinct monochromatic-blocks of same colour, and consider vertices $v \in B$ and $v' \in B'$. Notice that $d(v, v') \geq k+1$, since every two distinct blocks with same colour has at least one block of size equal to k with distinct colour between them. Thus, a monochromatic P_3 must be contained in a monochromatic-block, a contradiction. $\square$

Please refer to Fig. 5 for an example of a graph that achieves the upper bound.

H Proof of Theorem 5

Proof of Theorem 5. Let $G = C_n^k$, $2k+2 \leq n \leq 3k+1$, be a power of a cycle. We define $\pi : V(G) \to \{blue, red\}$ as follows: a monochromatic-block of size k with colour *red* followed by a monochromatic-block of size $n-k$ with colour *blue*. We refer to Fig. 6 to illustrate the given 2-biclique-colouring.

Recall that every biclique in G is an induced C_4. Now, we prove that every induced C_4 is polychromatic by the given colouring. Suppose, by contradiction, that there exists a monochromatic C_4 and denote it by H. If H is contained in a block of size k, then it is a subgraph of a K_4. Otherwise, subgraph H must be contained in a monochromatic-block of size at most $2k + 1$, since $n \leq 3k + 1$.

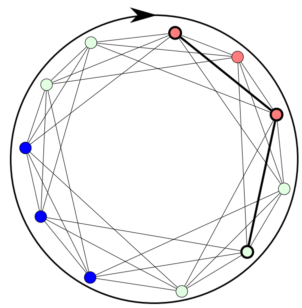

Fig. 5. Power of a cycle C_{11}^3 with biclique-chromatic number 3. Notice that C_{11}^3 has a P_3 biclique of reach 4 highlighted in bold; and also a square biclique.

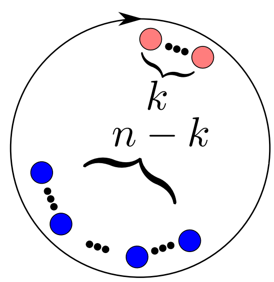

Fig. 6. 2-biclique-colouring of powers of cycles, when $2k+2 \leq n \leq 3k+1$.

Now, let B and B' be the monochromatic-blocks of colours *blue* and *red*, respectively. Consider the cyclic order of G and let vertex $u \in B$ be the last vertex of H before a vertex $v \in B'$ and vertex $w \in B$ be the first vertex of H after a vertex $v \in B'$. Note that u and w are adjacent in H. The right reach $\overrightarrow{d(u,w)}$ is at least $k+1$, because block B' has size k. Thus, the left reach $\overleftarrow{d(u,w)}$ is at most k. Since the other vertices of H have their indexes lying between the indexes of u and w, subgraph H is contained in a K_4, a contradiction. $\square$

I Proof of Lemma 3

Proof of Lemma 3. First, suppose that all the monochromatic-blocks have size k or $k+1$. Each of the edges of G is of one of the two following types:

- an edge between two vertices belonging to the same block (and having the same colour); or
- an edge between two vertices belonging to consecutive blocks (and having the distinct colours).

Hence, if we consider any three vertices v_i, v_j and v_ℓ having the same colour, then either they are in the same block — and induce a triangle — or they belong to at

least two non-consecutive blocks – and induce a disconnected graph. In neither case, they can induce a P_3 and, in particular, a P_3 of reach at most $k + 2$.

Now, suppose that there exists a monochromatic-block whose size is neither k nor $k + 1$. If a block has size $p \geq k + 2$, say composed of vertices $\{v_i, v_{i+1}, v_{i+2}, \ldots, v_{i+k+1}, \ldots, v_{i+p-1}\}$, then vertices $\{v_i, v_{i+1}, v_{i+k+1}\}$ induce a P_3. So, we may assume that there exists a block coloured blue with $k-x$ vertices, $x > 0$. We denote by v_i the leftmost vertex of the block — hence the rightmost vertex is $v_{i+k-x-1}$. Note that vertices v_{i-1} and v_{i+k-x} are adjacent and coloured red. Please refer to Figure 3. We consider the following cases.

- vertex v_{i+k} is coloured red. In this case, vertices v_{i-1}, v_{i+k-x} and v_{i+k} induce a monochromatic P_3 (note that vertex v_{i+k} is not adjacent to vertex v_{i-1}) with reach $k + 1$. This case is depicted in Fig. 3a.
- vertex v_{i+k} is coloured blue. We consider vertex v_{i+k+1} and two subcases.
  - vertex v_{i+k+1} is coloured blue. In this case, vertices v_{i+k}, v_{i+k+1} and v_i induce a monochromatic P_3 (note that vertex v_{i+k+1} is not adjacent to vertex v_i) with reach $k + 1$. This case is depicted in Fig. 3b.
  - vertex v_{i+k+1} is coloured red. In this case, vertices v_{i-1}, v_{i+k-x} and v_{i+k+1} induce a monochromatic P_3 (note that vertex v_{i+k+1} is not adjacent to vertex v_{i-1}, but is adjacent to vertex v_{i+k-x} because $x < k$) with reach $k + 2$. This case is depicted in Fig. 3c.

□

J Proof of Theorem 6

Proof of Theorem 6. Let $G = C_n^k$, $n \geq 3k + 2$, be a power of a cycle and let $\pi : V(G) \to \{blue, red\}$ be a 2-colouring of the vertices of G. By Lemma 3, graph G has **no** monochromatic induced P_3 if and only if every monochromatic-block has size k or $k + 1$. We refer to Fig. 2d to illustrate this 2-biclique-colouring.

Suppose every monochromatic-block has size k or $k + 1$. By Lemma 3, there is a 2-colouring of the vertices of G, say π, that assures G has **no** monochromatic induced P_3. Thus, every biclique is polychromatic, since every biclique contains an induced P_3 and every induced P_3 is polychromatic. Then, the 2-colouring π is a 2-biclique-colouring.

On the other hand, suppose that not every monochromatic-block has size k or $k+1$. By Lemma 3, every 2-colouring of the vertices of G has monochromatic induced P_3 with reach at most $k + 2$. Since $n \geq 3k + 2$, this P_3 is not contained in any induced C_4. Then, every 2-colouring of G is not a 2-biclique-colouring.

To conclude the proof, let a (resp. b) be the number of monochromatic-blocks with size k (resp. $k + 1$). Then, $a + b \geq 2$ because we have at least one block for each colour and $a + b$ is even because it is the number of monochromatic-blocks and every monochromatic-block is followed by a monochromatic-block with the other colour, i.e. the number of monochromatic-blocks with colour blue is the same as the number of monochromatic-blocks with colour red. □

K Computing the biclique-chromatic number of C_n^k, when $n \geq 3k + 2$

Theorem 9. *There exists an algorithm that computes the biclique-chromatic number of a power of a cycle C_n^k, when $n \geq 3k + 2$.*

Proof of Theorem 9. We rewrite the equality $n = ak + b(k + 1)$ in a very similar way to the division algorithm, but there is a rather subtle difference, since, in the new form, the choice for the value of the quotient depends on the choice of the value for the remainder.

By Theorem 6, a power of a cycle C_n^k, when $n \geq 3k+2$, has biclique-chromatic number 2 if and only if there exist two integers a and b, such that $n = ak+b(k+1)$ and $a+b$ is even. Otherwise, by Theorem 4, it has biclique-chromatic number 3.

Denote $c := a+b$. Then, $n = ak+b(k+1) = ak+bk+b = (a+b)k+b = ck+b$. W.l.o.g, suppose $0 \leq b < 2k$. If $b \geq 2k$, we can repeatedly replace (in the equality of n) c by $c + 2$ and b by $b - 2k$ until $b < 2k$, such that the "new" c still even, the "new" b still positive and the equality of n holds. If $b < 0$, it is analogous. We have two cases.

- $0 \leq b < k$. Then, $c = \left\lfloor \frac{n}{k} \right\rfloor$ and $b = n - ck$.
- $k \leq b < 2k$. Then, $c = \left\lfloor \frac{n}{k} \right\rfloor - 1$ and $b = n - ck$.

Now, to decide if the biclique-chromatic number of C_n^k is 2, we should check if the given solution has $a \geq 0$, $b \geq 0$, and even $a+b \geq 2$. By the above calculus, $b \geq 0$. Then, $a \geq 0 \Leftrightarrow a + b \geq b \Leftrightarrow c \geq b$. If $c \geq b$ and c is even, the biclique-chromatic number of C_n^k is 2. Otherwise, it is 3. $\square$

L Graphics of κ_B as function of number of vertices

In Figs. 7 and 8 we illustrate the biclique-chromatic number for fixed value of k and increasing n of powers of paths and powers of cycles, respectively.

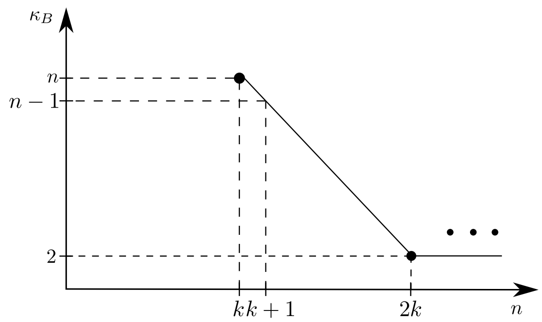

Fig. 7. The biclique-chromatic number of a power of a path for fixed value of k and increasing n

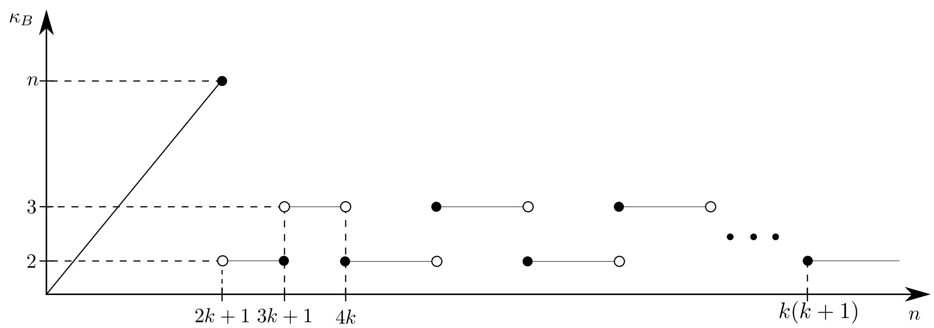

Fig. 8. The biclique-chromatic number of a power of a cycle for fixed value of k and increasing n

M Algorithms

Algorithm 1: To compute the biclique-chromatic number of a power of a cycle C_n^k with $n \geq 3k+2$

---

input : C_n^k, a power of a cycle with $n \geq 3k+2$
output: $\kappa_B(C_n^k)$, the biclique-chromatic number of C_n^k.

begin
  $c \longleftarrow \lfloor \frac{n}{k} \rfloor$;
  $b \longleftarrow n - ck$;
  if $c \mod 2 = 0$ *and* $c \geq b$ **then**
    return 2;
  else
    $c \longleftarrow \lfloor \frac{n}{k} \rfloor - 1$;
    $b \longleftarrow n - ck$;
    if $c \mod 2 = 0$ *and* $c \geq b$ **then**
      return 2;
    else
      return 3;

Algorithm 2: To compute a 2-biclique-colouring of a power of a cycle C_n^k with $2k+2 \leq n \leq 3k+1$

---

input : C_n^k, a power of a cycle with $2k+1 \leq n \leq 3k+1$
output: π, a 2-biclique-colouring of C_n^k.

begin
  vertex $\longleftarrow 0$;
  for $i = 1$ **to** k **do**
    $\pi(vertex) \longleftarrow 1$;
    $vertex \longleftarrow vertex + 1$;
  for $i = k+1$ **to** n **do**
    $\pi(vertex) \longleftarrow 1$;
    $vertex \longleftarrow vertex + 1$;
  return π;

Algorithm 3: To compute a 2-biclique-colouring of a power of a cycle C_n^k with $n \geq 3k+2$

input : C_n^k, a power of a cycle with $n \geq 3k+2$
output: π, a 2-biclique-colouring of C_n^k.

```
begin
    a ⟵ 0;
    b ⟵ 0;
    c ⟵ ⌊n/k⌋;
    b ⟵ n − ck;
    if c mod 2 = 0 and c ≥ b then
        a ⟵ c − b;
    else
        c ⟵ ⌊n/k⌋ − 1;
        b ⟵ n − ck;
        if c mod 2 = 0 and c ≥ n − ck then
            a ⟵ c − b;
        else
            return error;
    vertex ⟵ 0;
    for i = 1 to a do
        for j = 1 to k do
            π(vertex) ⟵ i mod 2;
            vertex ⟵ vertex + 1;
    for i = a + 1 to a + b do
        for j = 1 to k + 1 do
            π(vertex) ⟵ i mod 2;
            vertex ⟵ vertex + 1;
    return π;
```

Algorithm 4: To compute a 3-biclique-colouring of a power of a cycle C_n^k with $n \geq 2k+2$

```
input  : C_n^k, a power of a cycle with n > 2k + 1
output: π, a 3-biclique-colouring of C_n^k.
begin
    a ⟵ 0;
    r ⟵ 0;
    r' ⟵ 0;
    if n mod k = 0 then
        if n/k mod 2 = 0 then
            a ⟵ n/k;
            r ⟵ 0;
        else
            a ⟵ n/k − 1;
            r ⟵ k;
    else
        x ⟵ ⌊n/k⌋;
        r' ⟵ n mod k;
        if x mod 2 = 0 then
            a ⟵ x;
            r ⟵ 0;
        else
            a ⟵ x − 1;
            r ⟵ k;
    vertex ⟵ 0;
    for i = 1 to r do
        π(vertex) ⟵ 2;
        vertex ⟵ vertex + 1;
    for i = 1 to k do
        π(vertex) ⟵ 1;
        vertex ⟵ vertex + 1;
    for i = 1 to r' do
        π(vertex) ⟵ 2;
        vertex ⟵ vertex + 1;
    for i = 2 to a do
        for j = 1 to k do
            π(vertex) ⟵ i mod 2;
            vertex ⟵ vertex + 1;
    return π;
```